9.1 In this exercise, we will establish *Birkhoff's theorem* for spherically symmetric solutions to the vacuum Einstein equations in $3+1$-dimensions.

(a) Let $(\mathcal{M}^{3+1}, g)$ be a Lorentzian manifold such that $\mathcal{M} = \mathcal{Q}^{1+1} \times \mathbb{S}^2$ and, in any local coordinate chart (x^0, x^1) on $\mathcal{Q}$ and using the standard (θ, ϕ) coordinates on $\mathbb{S}^2$, g takes the form

$$g = \tilde{g}_{AB} dx^A dx^B + r^2 \big(d\theta^2 + \sin^2\theta d\phi^2 \big)$$

with $A, B \in \{0, 1\}$ and:

* $\tilde{g}_{AB}$ and r depend only on x^0, x^1,
* $r > 0$.

Deduce that $(\mathcal{M}, g)$ is spherically symetric, i.e. $SO(3)$ acts isometrically on $(\mathcal{M}, g)$ with spherical orbits. Show also that, around any point $p \in \mathcal{Q}$, there exists a local coordinate system (u, v, θ, ϕ) around $\{p\} \times \mathbb{S}^2$ such that

$$g = -\Omega^2(u,v) du dv + r^2(u,v) \big(d\theta^2 + \sin^2\theta d\phi^2 \big).$$

(such a coordinate system is called *double null*). *Hint: Use Exercise 2.3.*

Remark. It can be shown that any spherically symmetric spacetime can be expressed locally in the above form.

(b) Assume that $(\mathcal{M}, g)$ above satisfies the vacuum Einstein equations $Ric_{\alpha\beta} = 0$. In double null coordinates, it can be easily calculated that this system of equations takes the following form in terms of the metric components Ω and r:

$$\begin{aligned}
\partial_u \partial_v (r^2) &= -\frac{1}{2}\Omega^2, \\
\partial_u \partial_v \log(\Omega^2) &= \frac{\Omega^2}{2r^2} \big(1 + 4\Omega^{-2} \partial_u r \partial_v r \big), \\
\partial_u (\Omega^{-2} \partial_u r) &= 0, \\
\partial_v (\Omega^{-2} \partial_v r) &= 0.
\end{aligned}$$

(note that this is an overdetermined system; this is why, at the end of the day, Birkhoff's theorem holds). Show that the quantity $m : \mathcal{Q} \to \mathbb{R}$ defined by

$$m \doteq \frac{r}{2} \big(1 - g^{\alpha\beta} \partial_\alpha r \partial_\beta r \big) = \frac{r}{2} \big(1 + 4\Omega^{-2} \partial_u r \partial_v r \big)$$

(which is known as the *Hawking mass* of the sphere $\{p\} \times \mathbb{S}^2$) is locally constant on $\mathcal{Q}$.

(c) Let g_M be the Schwarzschild metric for $M \in \mathbb{R}$. Show that, in this case, $m = M$.

(d) Let $p \in \mathcal{Q}$ and assume, without loss of generality, that $(u(p), v(p)) = 0$. Show that there exists an open neighborhood $\mathcal{U}$ of $\{p\} \times \mathbb{S}^2$ in $\mathcal{M}$ and an open neighborhood $\mathcal{U}_{Sch}$ of a point q in the maximally extended Schwarzschild spacetime with $M = m(p)$ (chosen so that $r(q) = r(p)$) which are isometric. *Hint: Choose coordinates u, v on $\mathcal{U}_{Sch}$ so that the functions $\partial_u r(u, 0)$ and $\partial_v r(0, v)$ are the same in both spacetime domains. Deduce that the functions $r(u, v)$ and $\Omega(u, v)$ are the same for both spacetime domains, using the system of equations.*

9.2 Let $(\mathcal{M}, g)$ be a Lorentzian manifold and $S \subset \mathcal{M}$ be a submanifold. For any vector field W along S which is orthogonal to S, we will define the associated second fundamental form $\chi^{(W)} : \Gamma(S) \times \Gamma(S) \to \mathbb{R}$ by the relation

$$\chi^{(W)}(X, Y) \doteq g(\nabla_X W, Y),$$

where ∇ denotes the connection of g and we think of X, Y as being extended to vector fields in $\mathcal{M}$.

(a) Show that $\chi^{(W)}$ is well defined independently of the choice of extensions of X, Y. Show also that it is a symmetric $(0, 2)$-tensor field.

(*b) Assume that S is *spacelike*; we will also denote the induced (Riemannian) metric on S by h. Let W be a non-vanishing vector field on $\mathcal{M}$ which is orthogonal to S and let $\Phi_t^{(W)}$ be the flow map of W. For the one parameter family of surfaces $S_t = \Phi_t^{(W)}(S)$, with induced metrics h_t, show that, in any coordinate chart (x^1, x^2) on $S_t$which is transported along the flow of W:

$$\frac{d}{dt}\sqrt{\det(h_t)}\Big|_{t=0} = \mathrm{tr}_h \chi^{(W)} \cdot \sqrt{\det(h)},$$

where $\mathrm{tr}_h \chi^{(W)} \doteq h^{AB}\chi^{(W)}_{AB}$ For this reason, $\mathrm{tr}_h \chi^{(W)}$ is usually called the *expansion* in the direction of W, since it measures the rate of change of the volume form of S. *(Hint: You might want to use Jacobi's formula from linear algebra: $\frac{d}{dt}\log(\det M) = \mathrm{tr}(M^{-1}\frac{d}{dt}M)$ for a square-matrix valued function $M(t)$.)*

(c) We will now restrict to the case when M is $3+1$ dimensional and time oriented and that S is a 2-dimensional surface.. in that case, at each point $p \in S$, the normal bundle $TS^{\perp}$ is spanned by two **future directed null** vector fields along S, which we will denote with L and $\underline{L}$. We will also denote the induced (Riemannian) metric on S by h. We will say that such a surface S is **trapped** if it is compact and, at every point on S, both null expansions are negative, i.e.

$$\mathrm{tr}_h \chi^{(L)}, \mathrm{tr}_h \chi^{(\underline{L})} < 0.$$

Show that, on the maximally extended Schwarzschild spacetime, the spheres of symmetry are trapped if and only if they correspond to points in the region II of the Penrose diagram (i.e. the black hole region).

Remark. We will later see in class that, as a consequence of Penrose's incompleteness theorem, if an asymptotically flat spacetime contains a trapped surface S, then this is necessarily inside a black hole, i.e. $J^+[S]$ does not reach future null infinity $\mathscr{I}^+$. Since the condition defining a trapped surface is anopen condition, a trapped surface remains trapped even under small changes of the metric; thus, small perturbations of Schwarzschild spacetime still contain a black hole.

9.3 Let

$$T_{\mu\nu}[\phi] = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_{\beta\phi}$$

be the energy momentum tensor associated to the scalar wave equation $\Box_g\phi = 0$ on $(\mathcal{M}, g)$ (recall that $\Box_g \doteq g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = \frac{1}{\sqrt{-\det g}}\partial_\alpha\big(\sqrt{-\det g}g^{\alpha\beta}\partial_\beta\big)\big)$. For the first two questions, we will not assume that $\phi : \mathcal{M} \to \mathbb{R}$ solves any particular equation.

(a) Show that, for any $\phi \in C^\infty(\mathcal{M})$, any $p \in \mathcal{M}$ and any two future oriented causal vectors $V, W \in T_p\mathcal{M}$:

$$T_{\mu\nu}[\phi]V^\mu W^\nu \geqslant 0$$

(*Hint: Choose a suitable double null frame in* $T_p\mathcal{M}$). If V, W are moreover timelike, show that

$$T_{\mu\nu}[\phi]V^\mu W^\nu \geqslant c\sum_{i=0}^{n}|\partial_i\phi|^2,$$

with the constant $c > 0$ depending on V, W, g and the choice of local coordinates (but is independent of ϕ).

(b) Assume, now, that ϕ solves $\Box_g\phi = 0$. Show that

$$(\mathrm{div}T[\phi])_\nu \doteq g^{\alpha\beta}\nabla_\alpha T_{\beta\nu}[\phi] = 0.$$

(c) Show that, if, in addition, V is a Killing vector field of $(\mathcal{M}, g)$, then the 1-form $J^V_\nu[\phi] \doteq T_{\mu\nu}[\phi]V^\mu$ is divergence free, i.e.

$$\mathrm{div}J^V[\phi] \doteq g^{\alpha\beta}\nabla_\alpha J^V_\beta[\phi] = 0.$$

9.4 Let $f : [0, T] \to [0, +\infty)$ satisfy

$$f(t) \leqslant A(t) + \int_0^t M(s)f(s)\,ds$$

for some non-negative functions A, M on $[0, T]$. Show that

$$f(t) \leqslant A(t) + \int_0^t e^{\int_s^t M(x)\,dx}M(s)A(s)\,ds.$$

In particular, if $A(t) = A$ is constant, show that

$$f(t) \leqslant e^{\int_0^t M(s)\,ds}A.$$

This is known as *Gronwall's inequality*; this inequality will play a crucial role in establishing energy-type estimates for hyperpolic PDEs. (*Hint: You might want to first consider the differential inequality satisfied by $F'(t)$ for $F(t)$ being the right hand side of the inequality we start with.*)